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Mathematical Preparation for Secondary-Level Mathematics Teachers: Interweaving Perspectives of a Mathematician and a Mathematics Teacher Educator

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Context

This paper is a translated and enhanced version of a book chapter outlining elements of collaboration between a mathematician and a mathematics teacher educator on issues related to the mathematical preparation of secondary-level mathematics teachers.

Introduction

The interaction that we describe between a mathematician and a mathematics teacher educator takes its source from an invitation to Frédéric to give a talk at a conference about issues of the mathematical preparation of mathematics teachers (Proulx, Corriveau, & Squalli, 2012). Frédéric was invited to give a plenary talk on the basis of the innovative work that he and his colleagues have been conducting for a number of years at the Université Laval in mathematics courses for future secondary-level mathematics teachers (Hodgson, 2001; Gourdeau, 2010, 2015). Because he had in the past delivered these kinds of addresses in various venues (for mathematicians, for didacticians, for teachers, etc.), Frédéric's response to the invitation was to return the challenge. Aiming more than only to "explain" his work so that the experience would make him grow and even question his own work, he boldly suggested turning the invitation into an occasion for collaboration with a mathematics educator, to discuss deep questions, perspectives, research issues, and outcomes about teachers' mathematical education. In this context, Jérôme as a

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mathematics education researcher was "inserted" into the invitation-collaboration endeavor in order to nourish the talk that Frédéric would deliver at the conference. The following is thus the concise result of this collaboration, where Jérôme visited Frédéric to attend a number of his mathematics courses and engage in a variety of discussions on these matters. Along the way, other colleagues joined the collaboration, particularly Bernard Hodgson, who has worked with Frédéric for a number of years and participated in the elaboration of their program's orientations, and Jean-François Maheux, Jérôme's colleague at UQAM.

Thus this article is a translated and reshaped/enhanced version of the work produced for the conference that appears as a book chapter in French (Gourdeau & Proulx et al., 2012). This endeavor of producing an English-language version of our work was motivated by a somewhat flattering suggestion from Nadine Bednarz, in her address at the 40th anniversary of the Canadian Mathematics Education Study Group (CMESG), that this work would be of interest to the English-speaking community.[1] After setting up the context of the program in which Frédéric offers his mathematics courses, the following takes the form of a *question and answer* interaction between Frédéric and Jérôme, to which an insertion is made by Jean-François toward the end. This is concluded with a retrospective view on the collaboration five years later.

Context: the secondary-level mathematics teachers' preparation in the mathematics department at Université Laval

The four-year undergraduate program (B.Ed.) for secondary-level mathematics teachers includes 15 compulsory courses in mathematics, statistics, and computer science. Six of these are designed explicitly for secondary-level mathematics teachers: these courses are called *teacher-specific* in contrast to *basic courses. Teacher-specific* courses, given exclusively to students in the B.Ed. program, are taken throughout the four-year program, starting in the third semester, and are interwoven with other education-related courses[2] and practicum. Five are in mathematics and one in statistics.

Before engaging in these teacher-specific mathematics courses, students take seven basic courses in mathematics, one course in statistics, and one in computer science. These basic courses are taken with students from other programs, mostly from the undergraduate program in mathematics. This enables students in the B.Ed. program to develop stronger foundations in mathematics while negotiating the transition to university. The team at Laval believes that students are then better prepared to engage in issues related to secondary-level mathematics teaching, including cultural dimensions, which are addressed in *teacher-specific* mathematics and statistics courses.

The following is an outline of the 15 courses in mathematics, statistics, and computer science that are part of the B.Ed. program.

Basic mathematics courses

MAT-1200	Introduction to linear algebra (*1st semester*)
MAT-1110	Multivariate calculus (*1st semester*)
MAT-1300	Elements of mathematics (*1st semester*)
MAT-1100	Analysis I (*2nd semester*)
IFT-1701	Introductory programming (*3rd semester*)

Other basic mathematics course on themes closely-related to secondary-level mathematics

MAT-1500	Geometry (*1st semester*)
MAT-1310	Discrete Mathematics (*2nd semester*)
MAT-2310	Number Theory (*3rd semester*)
STT-1000	Probability and Statistics (*4th semester*)

[1] The authors wish to thank the *Presses de l'Université du Québec* for allowing them to rework the chapter and publish this reshaped version in *Chroniques*.

[2] In particular, students have to take the three mathematics education courses of the program, namely *Didactique des mathématiques* I, II and III: DID-2030, DID-2031, and DID-3030.

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Teacher-specific mathematics courses

MAT-2904	Analysis – a complement (3rd *semester*)
MAT-2906	Fundamental Mathematics for Teaching (4th)
MAT-2901	Mathematics and Technology (4th *semester*)
MAT-2903	Mathematical Topics for Teaching (5th)
STT-2902	Statistical Modelling (5th *semester)*
MAT-3900	Evolution of Ideas in Mathematics (8th)

Initiating the interaction with a question

It is in this earlier context that the collaboration between Frédéric and Jérôme took form and was grounded. It took the shape of interaction around a specific set of questions triggered by the conference organizing committee, which aimed for Frédéric to address the following questions during his plenary address at the conference.

> *Beyond ministerial prescriptions, what is the rationale for having secondary-level mathematics teachers take advanced or basic mathematics courses in their teacher preparation? As well, what is the rationale for having designed teacher-specific mathematics courses for future secondary-level mathematics teachers?*

What follows is how Frédéric and Jérôme interacted with each other about these questions, which took the form of a dialogue.

Frédéric's answer to the initial question

Secondary-level mathematics teachers have a complex task to accomplish, and they are alone and in charge of their classroom in all their teachings. To be able to give their full attention to students and their understanding, they need to have a profound and solid understanding of the mathematics embedded in the secondary school curriculum. They need to be able to choose explanations given to students, to build learning activities, to choose which aspects should not be brought to the fore (e.g., some of the difficulties with irrational numbers, which arise in many varied contexts) and which should be explained, in part or in full.

Insufficient understanding of mathematics can lead to rigid, inflexible, or even poorly prepared teaching at the mathematical level, reducing it to a series of algorithms to master. It can also lead to an implicit or explicit acceptance of erroneous reasoning or understanding, such as might arise when a teacher does not know what to do with unexpected approaches or explanations.

Standard courses in mathematics partly address this need. In the best of scenarios, after adequate mathematical preparation in standard mathematics courses, the future teacher will understand advanced mathematics to a high degree and will be able to explain its usefulness in mathematics and for applications. This teacher will also have an excellent mastery of algorithmic mathematics and will be able to solve complex problems using his or her mathematical knowledge.

However, this personal efficiency in using mathematics will probably have been gained using efficient mathematical language that is not generally accessible to secondary-level students. Teachers will have to manage the transition between their own understanding of mathematics and their secondary-level students' learning of mathematics. Furthermore, because the advanced mathematics courses are not aimed (solely) at educating teachers, future teachers will not have been taught with the (partial) intent that they should understand why a mathematical notion may be difficult, but rather in a way that directly aims at learning without facing these difficulties (as much as possible). This is a first indication that *teacher-specific* courses can play a role.

Here is an example to illustrate these ideas. In one of the *teacher-specific* mathematics courses in which I intervene (MAT-2906 Fundamental Mathematics for Teaching), students have a significant problem-solving project. A seemingly almost identical project is undertaken as part of a problem-solving course for mathematics majors at the same university. The key difference is that the main objectives of these two projects differ substantially. For mathematics

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majors, the essential goal is that students become better at mathematical problem-solving: this is the main motivation for students, and the work is tailored to achieve this as much as possible. For pre-service teachers, the emphasis is on the understanding of problem-solving processes (for themselves and others), of their complexity, their phases and processes, including affective dimensions, with the overall goal of being able to work afterward with secondary-level students. Thus in these two courses, which I have personally taught to both groups of students, a problem-solving diary is part of the work on problem-solving, but the nature of the work each course entails is significantly different.

Furthermore, problem-solving over an extended period will allow students in the B.Ed. program to learn about genuine mathematical activity through living it. They will be able to deepen their understanding of what mathematics is by working creatively in mathematics for many hours, and even days or weeks. They will better understand and appreciate the roles of exemplification, generalizations, conjectures, and much more.

Jérôme's answer to the first question

I think we share similar visions of the *mathematical* complexities of the teacher's task, where the teacher needs to be well equipped to undertake it. But let's get to the main issues. You mention:

> To be able to give his full attention to students and their understanding, [teachers] need to have a profound and solid understanding of the mathematics embedded in the secondary school curriculum. They need to be able to choose explanations given to students, to build learning activities, to choose which aspects should not be brought up and which should be explained in part or in full.

I have always wrestled with the following question, which could seem naïve: *In what ways are advanced mathematics courses really helpful in this endeavor?*

My visits in some of your advanced and teacher-specific mathematics courses have triggered numerous reflections in me. So let me take a stab at this question in relation to what I have observed in your courses, but also in relation to some aspects of the research literature that I am aware of and that can contribute to the discussion.

I start with the literature. For a number of years, advanced mathematics courses have been questioned, to say the least, as a means of mathematically educating future secondary school mathematics teachers. First, for a long time, questions have been raised on the relevance of these courses, some positively and others less so.

> The question of what mathematical training a future mathematics teacher should receive produced divergent views. There were those who advocated 'mathematics for teachers' with emphasis on those aspects of mathematics which featured in school courses; others pointed to the position of mathematics teachers as members of the mathematical community and felt that two different types of mathematics do not exist. This is likely to remain an unresolved question, but one that required continuous examination. (Otte, 1976, pp. 200-201)

Hence numerous arguments have been stated, but also criticisms, for example, about the formal, symbolic, and compact nature of the content being worked on in these advanced mathematics courses (see, e.g., our review in Proulx & Bednarz, 2008). However, it seems to me even more relevant to reflect on concerns about how mathematics is being used in these courses and also how a privileged form of teaching is being used.

Thus one frequently reported criticism of advanced mathematics courses is about how these courses are delivered. As Bauersfeld (1994) and Burton (2004) have reported, the usual way of delivering these courses is using a lecture style where mathematical concepts, definitions, theorems,

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axioms, proofs, and so forth, are exposed (or "broadcasted," as Cooney, 1988, would say). Mathematical habits and systems of the culture in which students are thus immersed relates more to a reified knowledge base to incorporate, than to what Brousseau (1998) calls an active participative process of mathematization.[3] So there is a significant cultural rupture or distance between these ways of doing mathematics and what is promoted for mathematics teaching in secondary classrooms. According to Bauersfeld, teachers need to be immersed in a mathematical culture, in mathematical practices where mathematics is worked on and is vivid and evolving, in contrast to being introduced as an allegedly objective and fixed body of knowledge where mathematics is perceived as an epistemological absolute.

Burton's (2004) study of mathematicians is interesting in this regard. Through an analysis of mathematicians' discourse about their practice, her study presents important differences between their cultures of teaching and of researching mathematics. Many of the mathematicians whom she interviewed underlined, ironically, the incoherence existing between their professional practices when they *do* mathematics and their teaching practice centered on exposing mathematical concepts, the latter offering a view of mathematics as a preexisting and static body of knowledge in contrast to the former where mathematics is lived as a continually evolving discipline requiring continual personal constructions for the one doing them. Hence mathematicians themselves assert that their teaching practices in advanced university mathematics courses are in important rupture with how mathematics is construed and produced. This distance can question the relevance of these courses for educating teachers, reinforcing the argument that advanced mathematics courses appear not to prepare future teachers adequately for the authentic processes of construction of mathematics, to how mathematics is developed: and thus poorly contributes to what are termed flexible practices in teachers.

However, I could not relate this criticism to your own teachings and courses: far from it in fact. One of the many aspects that I have found interesting in the advanced as well as teacher-specific mathematics courses that I have attended is about the notion of "doing mathematics." However, I do not mean "doing mathematics" just because "students were doing mathematics in your courses". I mostly mean that there was a discourse on "how mathematics is done." Let me explain. Krummheuer (1992) underlined the importance of teachers' argumentation formats to orient how students argue and do mathematics in the classroom. For example, he showed how teachers' mathematical practices in their classrooms (through the examples and explanations they gave, their ways of justifying, etc.) were communicating, often implicitly, how mathematics should be done in their classrooms. I experienced this issue in your courses at the implicit, but also the explicit level, where you took time to explain to students how mathematics was done (i.e., what one could allow oneself to do and what could not be done). Perhaps this looks obvious to you, but I felt an underlying richness in this practice, one that invited students into an authentic mathematical culture.

For example, you addressed special cases that one could set aside to continue working on a general proof (sometimes coming back to it, or not). You also discussed definitions that needed to be tweaked or even biased to adapt them better to the situation being studied, as well as highlighted the potential of inventing values in a problem in order to start solving

[3] Obviously not all advanced mathematics courses fit this model/critique, and we know of people who do/did differently (e.g., to name a few: Whiteley at York University, Legrand in Grenoble, Moore method in the USA).

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it and adapting afterward. These ideas sharply contrasted with naïve or romantic views of what mathematics is, as Davis and Hersh (1981) or Lakoff and Nuñez (2000) would say, where mathematics is conceived of as a rigid and pre-formulated science that follows clearly outlined and established protocols of conduct. In short, in your practices, one can feel the significance of what Turkle and Papert (1993), referring to Levi-Strauss (1967), call *bricolage* practices. Bricolage cannot be reduced to the application of preformulated tools: it is an expertise in action, the combination and adaptation of earlier experiences to address current problems (another aspect that Burton's (2004) mathematicians outline as being representative of a mathematician's problem-solving expertise).

Also, I found it interesting that you were often positioning yourself in rupture with the usual schooling practices, making distinctions between what I personally call "mathematics in schools" and "mathematics as a discipline." This provided an additional layer of vividness to the mathematical practices enacted in your courses. For example:

- You traced a difference between a symmetry and a reflection, two terms often treated as synonyms in secondary schools;
- You insisted on the difference between a geometrical construction and the justification of this same construction as being two different, but necessary practices;
- You established the difference between the definition of a concept and its results or formula as two connected but separate entities;
- You worked with generic examples to prove conjectures, contrasting this practice with the use of unique examples and general or algebraic proofs.

These practices seemed convincing reasons for having a professional and active mathematician who can explain how mathematics is done on a day-to-day basis as a teacher of teachers.

Frédéric's response to the "rationale" comment

You address issues of importance about the rationale of having mathematicians participate in the mathematical education of future secondary-level teachers. Let me address this issue as well, but through discussing the courses themselves.

What are some of the advantages of teacher-specific courses?

One of the advantages is the possibility of structuring courses according to other aspects than mathematical topics (or content). We can stress unifying themes or ideas, for example, reasoning or infinity; or a broad topic treated in many diverse ways or using diverse approaches (for example, conics). As an example, in a course placing strong emphasis on reasoning, some seemingly unrelated topics can be addressed because they feature an important aspect of mathematical reasoning. For instance, if we wish to explore the difference between the definition of a concept, the definition of a notation, and the proof of a result, then we can consider the confusion that arises when we ask students to prove that $a^0 = 1$ (is this something we can actually prove?), thus looking at exponentiation. Other topics can naturally be introduced in this way. Another example is the transition from rational numbers to all real numbers, present in many areas of mathematics at the secondary level. This can be addressed as a key mathematical process, stressing the roles of approximation and continuity. Note that this does not necessarily imply using all the epsilon arguments, which are a must in most curriculum for mathematics majors.

What are some of the pitfalls?

A major pitfall is the potential lack of coherence between mathematics methods courses[4] and those in mathematics (or statistics), even those designed as being *teacher-specific*. There are many reasons for such lack of coherence, which can come, for example, from the organizational structure (lecturers from other units or departments, lack of stability as lecturers may change frequently) and the academic culture (academic freedom sometimes condoning a high level of individualism), as well as explicit physical distance between lecture halls for diverse departments (implicitly meaning that the courses are themselves about very different—or distant—subjects as you mentioned to me once from your work with Elaine Simmt of the University of Alberta, see Proulx & Simmt, 2011).

This lack of coherence is partly visible to students. They are used to considering success in their studies as a series of discrete successes, each exam, project, or course considered in isolation. Whereas we all hope that this is not so in our respective programs, there seem to be few cases where lecturers of courses in mathematics and lecturers in mathematics methods work together to establish a common vision, and from shared general objectives work together to establish more precise objectives for a series of courses. It might be interesting to consider team-teaching across departments, but such a project is generally closer to Utopia than reality. In my experience, although not impossible, the exchanges have been limited to general objectives and did not venture further. As writing about this in general terms may well describe situations that vary considerably, I try to clarify this by presenting two examples that I find relevant in the context of our discussions.

Generic examples

During one of your visits, you reacted positively to one of my interventions in the geometry course, when I touched on the notion of *generic example* with my students. Discussing generic examples in a geometry class or in a teacher-specific course can be done easily, and in numerous mathematics courses: examples abound, and generic examples can be foregrounded while working on a number of varied mathematical topics. (In geometry, often a specific triangle represents all triangles; in graph theory, a specific graph, drawn on a board, may well represent all graphs of a certain type; and in number theory, reasoning about a specific arithmetic series can have similar qualities.) Is this useful to forthcoming mathematics methods courses? What reflection on this could students have started in a mathematics course so that it can be pursued in a mathematics methods course? What are some other aspects or types of mathematical reasoning that would be useful to foreground in mathematics courses for secondary-level teachers? To me, it seems important that lecturers of mathematics and mathematics educators talk about this.

Mathematical writing

Going from the consideration of particular examples to the formulation of general statements, and back, is done using varying degrees of mathematical symbolism. The capacity to go in both directions appears to me, as a mathematician, essential to teachers. How is this approached, if at all, in mathematics methods courses? In mathematics courses, how should we take into account what has been or what will be done in mathematics methods courses when we work on that aspect?

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[4] In Quebec, these courses are often called *cours de didactique des mathématiques*, but we use here the Anglophone usual terminology of *mathematics methods course* instead of translating the francophone expression.

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Jérôme's new question to Frédéric about potential interactions between courses

A second issue of interest in what you mention above about future teachers is that "*Teachers will have to manage the transition between their own understanding of mathematics and their secondary-level students' learning of mathematics*". I have been struggling with this issue for a while. Let me contextualize this.

In view of some research results from Pian (2000, cited in the Commission Kahane, 2003), where teachers struggle with this transition, or even through Klein's (1932) well-known notion of double-discontinuity for teachers between academic mathematics and schooling practices, I am wondering if it is not the role of teacher education training to intervene, or at least help and participate, in this transition. In short, should the weight of transition be left uniquely on future teachers' shoulders? And how can mathematics and mathematics methods courses interact and contribute to lifting that weight?

Some paths have been explored, if only in theory, in a working group in 2009 at the *Espace mathématique francophone* conference (Hache, Proulx & Sagayar, 2009). One of these ideas was to create/adapt advanced mathematics courses in which links between secondary mathematics and advanced mathematics concepts would be explicitly outlined. This is an idea that we discussed during some of my visits and for which you expressed interest. We also mentioned specific concepts and notions that could be used to reveal these links and affiliations: for example, for calculus courses, the rate of change and derivative notions, tangents, and tangents under the curve. In the working group in 2009, the idea for these courses was to work on advanced mathematics concepts, hence allowing teachers to know more about mathematics, and also enabling them to know how the concepts taught in school fit into the mathematical panorama and horizon (as some would call it, see e.g., Zazkis & Mamolo, 2011, 2012) and to establish links between these contents. One main rationale for this proposition was in fact to avoid leaving the weight of identifying these links on the shoulders of future teachers and to contribute to this process: to render explicit the links and articulation between secondary school and university mathematical concepts and to offer a view of where school mathematics concepts fit into the general mathematical panorama. Still, this leaves unanswered the place that mathematics methods courses can play in this dynamic.

The discussions we have had on these issues helped me to understand the links that you as a mathematician saw and deemed important. I recall a discussion we had with Bernard Hodgson during which he insisted on the fact that the mathematics that you decided to include in the teacher-specific courses needed to be related (closely or farther away) with the mathematics that future secondary teachers would be teaching. This led to a discussion of what could be meant by "be related," particularly in the function of being "closely or farther away." There was for me the idea that these teacher-specific courses did not work on mathematics for the sake of mathematics arbitrarily chosen, but because this mathematics needed to have some relevance for the future practices of teachers, hence in close or far affiliation with secondary school mathematics. But I also recall that this response was unsatisfactory when describing what happened in your courses, because at times it was not the mathematical concepts that were linked to school mathematics concepts, but mostly what was done with these concepts. In particular, you insisted on the idea of having future teachers "live" specific mathematical experiences and familiarizing them with some specific mathematical aspects that you deemed important. You also explained that what was important for you was that they had to live and be immersed in a "mathematical experience" of a certain nature in your courses (often independently of the mathematical content itself). Thus there was in your rationale the importance of having future

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teachers "living an authentic mathematical experience." Hence, it was not the content that was linked to school mathematics, but how mathematics and doing mathematics were experienced; a little like what I mentioned earlier about having teachers reflect on how mathematics could be done, what is allowed, how it is produced, thus what it means to do mathematics. So again, we come back to the idea of having future teachers experiencing mathematics, being immersed in engaging mathematical practices, of having teachers *do* mathematics (in contrast to only knowing it). Can you tell me more on these matters?

Frédéric's response to Jérôme's question on experiencing mathematical practices

Let me continue our discussion on three aspects. First, the discussion we engaged in about my claim that the teacher will have to do the transition between his understanding of mathematics and that of his students. I am certainly in agreement with what you wrote about this and would not wish to suggest that this work should be that of the teacher alone, and furthermore only once his or her preparation at university is completed. However, it appears to me that the personal mastery of mathematics is generally required for teaching. I also believe that mathematics methods courses must play a crucial role in the establishment of conditions that enable the students to learn mathematics.

I would like to go back to something you wrote earlier and that came out of a working group:

> We also mentioned specific concepts and notions that could be used to reveal these links and affiliations: for example, for calculus courses, the rate of change and derivative notions, tangents and tangents under the curve. In the working group in 2009, the idea for these courses was to work on advanced mathematics concepts, hence allowing teachers to know more about mathematics, and also enabling them to know how the concepts taught in school fit into the mathematical panorama and horizon and to establish links between these contents.

Adding these links is certainly always interesting. Furthermore, when this is done so as to enable pre-service teachers to understand how mathematics done at the tertiary level (college or university) originates from mathematical ideas or problems that are part of the secondary school curriculum, then I believe that we add significantly to their preparation as teachers. They can see and understand where questions and answers come from and better understand how a simple idea may in fact be the beginning of something deeper and broader: for example, using the difference between consecutive terms of a sequence to help find its general term, and trying to identify a function by studying its variation. Here, again, the collaboration between mathematics educators and mathematicians can be fruitful. *Which links or affiliations in the mathematical content of the secondary school are seen as more important from a didactical perspective?* The experts of the curriculum are not the mathematicians (at least not in Quebec). However, mathematicians may be well placed to perceive and exploit links with more advanced mathematics and help identify their importance beyond secondary school mathematics.

However, adding links is not sufficient if it is seen as only adding yet more content. I would like to see it being done in a way which enables students to "live a mathematical experience"; otherwise we may again have more facts to learn, poorly understood and soon forgotten. So here again, I am confronted with the importance of structuring courses so that students are themselves involved emotionally and intellectually in a mathematical endeavor. Beyond the magnificent mathematical world that has been established, and which we may wish for everyone to know, particularly teachers, there is a method of doing, of understanding, and of creating what we can call "mathematical." A personal experience of what this means, an experience and understanding that are as deep and profound as possible, and a personal

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mastery of this mathematical way of doing and understanding all seem important to me. For a teacher, being able to articulate coherently some of the more important aspects of this method of doing and understanding, for example like in *Thinking Mathematically* (Mason, Burton & Stacey, 1982) for aspects linked to problem-solving, strikes me as important.

Jérôme's additional comment on collaboration between mathematics and mathematics teacher education

The link between mathematicians and mathematics educators (or advanced mathematical courses and mathematics methods courses) is of interest to me (and are we not currently exactly doing this at some level?). I go back to a question to which I alluded earlier: *What can a professional mathematician offer by giving courses to future teachers?* I personally think that you, Frédéric, offer much to future teachers by being a professional mathematician who talks about mathematics (which is your object of study and work). I like to think of it in the same way that we find relevant to have actual practicing secondary school teachers implied in the education of future teachers, through courses, conferences, and so forth. They are professionals of mathematics teaching who talk about mathematics teaching (their object of work). Your colleague Bernard Hodgson has his own view of this and of how this is important (e.g., he argues about it in his ICME chapter in Hodgson, 2001). I tend to agree with him. And it is not for me a question that the professional mathematician offers *more* on the mathematical plan; it is not a question of better or worse. In the same way for the practicing teacher who takes a role in the education of teachers, for me, professional mathematicians offer and afford something *different*, something different from what mathematics teacher educators are offering and affording, by being people who produce mathematics in their everyday life. And I believe it is this difference that seems worth exploring and pursuing in the collaboration between mathematicians and mathematics teacher educators (and why not practicing teachers too?).

Inserted reaction from Jean-François

As you both adopted a personal tone, my reaction to your comments is rooted in my own experiences and reflections. Among these, two seem specially important here: (a) I have been a student in Frédéric's courses; and (b) I have subsequently continued my training to a doctorate, where I reflected a great deal about the links between the various people involved in mathematics education (especially researchers, teachers, and students—but also teacher educators whether they be mathematics educators or mathematicians!).

Reading your exchanges, I am struck by an aspect both fundamental and a little invisible, as obvious as it is. Your conversations have as a starting point a will on the part of Frédéric to reflect, to advance, and to question: his way of returning the invitation to the conference organizers for more intimate collaboration being a case in point! That same Frédéric, who has a long personal involvement and a sustained interest in the training of prospective teachers. This "personal" dimension seems to me to be of great importance, and yet somewhat obscured by the reflection that is developing around "training," "courses," "lived experiences," "skills to develop," "roles" of each other as "mathematicians" or "mathematics teacher educators," and so forth. We talk about training, courses, experiences, knowledge, and so on, but we are talking about *one* training in particular, a few courses designed and implemented by *one* particular person, who moreover turns out to be the very same person who wonders about how more advanced mathematics can serve (future) teachers.

I reiterate here one formidable idea that runs through your discussion: that of offering future teachers mathematical experiences of a particular quality. For me, it is about making them meet the mathematician and his or her work so that affiliations are established between their own work, that of their (future) students, and that of those whose profession it is to develop mathematics. As you both note so well, on the one hand, it seems good to have such experiences to develop a certain familiarity with mathematics, but also to make sure, on the other hand, that what these experiences trigger is made part of a "program" providing favorable conditions for exploring them. As an observer today, and as a former student of these courses, what strikes me nevertheless are the formidable ways of being of the people who design these courses and programs: the active presence of those with whom prospective teachers live these courses and programs.

I mean here that one cannot consider reducing these experiences to having mathematicians giving well-adapted courses in a well-thought-out program. The "well-adapted" and the "well-thought" are, above all things, taking shape in the actual encounter between the prospective teachers and the university instructor. That's the moment when they actually take shape and are transformed into a mathematical experience of a particular quality. In my case, it is the encounter with (a) Frédéric, his approach and nuances, his insightful pauses and his "you work on it" or his "but wait a little, it's not exactly *that*" played at the right moment. It is his ability to listen, to throw back, to be demanding, to see beyond, to reinterpret ideas. It is the encounter with all this wealth that made these courses mathematical experiences that give me the precious memory of tasting the magnificent fruit of which professional mathematicians make their daily bread.

In a nutshell, I realize (again?) that as interesting as are the questions of "what mathematics" to offer to future teachers, "how" to do so, for what "reasons," and with what "risks" or "benefits," we must not lose sight of the essential. This essential to me lies in meeting with a mathematician in the moment of living the mathematical activity. And for a simple reason: learning, knowing, training, and practicing one's profession (student, teacher, trainer, etc.) are first and foremost ways of being together: exactly as Frédéric did by coming to speak at that conference. Frédéric embodies that formidable example of a mathematician who comes to meet the other, whether he is a future teacher or a professional mathematics teacher educator.

Concluding remarks – five years after: foregrounding (hidden) issues, generating new issues

In the earlier book chapter (Gourdeau et al., 2012), the concluding remarks had been made hurriedly days before the actual conference and were mostly about how successful the collaboration had been and how each had grown and enriched himself through it. However, more than five years later, the exercise of translating, reworking, and enhancing this work revived the collaboration. In what follows, both Frédéric and Jérôme offer "five years after" reflections on the actual collaboration process, trying to make explicit what has potentially been gained from this collaboration (which is envisaged to be continued in the near future).

Frédéric's reflection some five years after, reading back…

Looking back on our reflections, I can see how working with Jérôme enabled me to become more aware of what I did, of what I could do, and possibly even of what I should do in some of my work with pre-service teachers. From his position as a researcher in mathematics education who was willing to engage in honest and thoughtful discussions, he provided insights that would not have been accessible to me otherwise, based as they were on both his observations and his knowledge. For example, through our discussions and written exchanges, I became aware of how important it could be to foreground some elements. Even if I did this regularly, I seldom did it as part of an explicit plan, and often not even in a deliberate manner. It was more an opportunity that I seized when I saw it, as is exemplified by the discussion about "generic examples" in the geometry class. This was not part of the syllabus.

In our discussions, I understood that for Jerôme, there was value in this. The value given by a mathematics teacher educator to some of these offhand remarks or short discussions was a surprise. Jean-François' comments, while focusing on human aspects, also drew my attention to this.

This move toward a more deliberate agenda, which goes beyond content and attitude, as well as the questions asked by Jerôme about the difficulty of the task faced by secondary school teachers, has led me to make a more concerted effort to create a working environment in the classroom that could be mirrored in secondary school teaching. It also led me to expand my knowledge of some areas of mathematics education.

These reflections have since been enriched by discussions with colleagues in Canada, including a visit to Simon Fraser University, where I was able to visit the secondary school classroom of Michael Pruner and observe a workshop led by Peter Liljedahl. I discussed with Peter the ideas introduced in his papers (Liljedahl, 2014, 2016), namely, the use of vertical non-permanent surfaces and random groupings. I have since adapted these in some of my courses, mostly in a problem-solving context.

I have also continued my journey, learning more about mathematics education as I continue to reflect and discuss. More recently, I have started making a more thorough attempt at teaching or creating learning activities with an explicit goal of understanding longitudinally, looking at topics in the mathematical curriculum that tie together aspects from the elementary/secondary grades onwards. This, I believe, also stems from fundamental questions asked by Jerôme.

A lot more could be said, but this may blur what I believe is the main point, the same that Nadine Bednarz highlighted at the CMESG-40 conference in 2016 that spurred the intention of producing this English-language version of the work: much can be gained from working together as mathematicians and mathematics teacher educators.

Jérôme's reflection some five years after, reading back…

The importance of the interaction between people, precise people and their actions, as Jean-François says… This idea reminds me of an aspect in Frédéric's initial concluding remarks in the earlier book chapter that the starting point of our interaction appeared simple: I attended courses in his program and then we discussed issues. But as simple as it seems, Frédéric reminded me that our first interactions struggled or even were confronted by our differing views, interests, and experiences about mathematics. Both curious about, and I add respectful of, the ideas of the other, we were clearly not always talking about the same mathematics. However, we rapidly went beyond usual ruptures and deciding who was right or wrong and discovered a mutual interest in the *doing of mathematics* (mostly in contrast to the presentation of reified mathematics). In short, we were interested in the classroom as a site for mathematical activity and to engage in these issues.

And five years later, what strikes me even more than it did then is the almost exclusive or fundamental focus or interest that we both seemed to share about mathematical activity: not necessarily setting aside content, but at the same time, yes. In many utterances we simply seemed to let it go.
This has for me been representative of the transition that I was implicitly undergoing as a mathematics teacher educator (but also, and maybe mainly, as a researcher). It occurred to me in this sense that what seems of utmost importance in the mathematical education of teachers, students, and society at large is what Papert (1972) calls *Mathematical Ways of Thinking* (MWOT). And I

believe, although often implicitly, that this is what Frédéric's teachings were about.

Papert (1972) suggests that in becoming a mathematician, one develops something more powerful than knowledge of content, which he calls MWOT. Doing mathematics enables one to develop particular ways of thinking that shape the mathematical mind of those who do it. This is again well aligned with Burton's (2004) studies, that I mention here and there in this article, which assert that mathematicians gain through the years some kind of experience to solve problems, experience that leads them to solve even more problems. For Papert, the development of MWOT is fundamental and enables the mastering of even more mathematical content. MWOT are a way of doing and functioning in mathematics, and these MWOT develop directly through doing mathematics. For Papert, contents are secondary to the mathematical activity, which is first and foremost influenced by MWOT. Hence I see MWOT somewhat as a bridge between mathematical content and practice: it is not about isolated practices or content: it is about practices, but practices about mathematics. As Papert (1972) would say, it is beyond only doing, and it is beyond only content: it is both, and together, intertwined.

However, all in all, reading back, our discussions are about future teachers who do mathematics. What about them as mathematics teachers? The second thing that reading this interaction five years later raised in me is about the execution or operationalization of these ideas by future mathematics teachers. We could think that Frédéric's university teaching, focused as it is on problem-solving and heavily on meaning-making and mathematical activity, is not the usual way mathematics is taught in most secondary school classrooms. So how can mathematics methods courses be part of the development of these practices in future teachers? What kinds of activities can be developed in mathematics methods courses *to build on* these experiences *for the mathematics classroom practices* of future teachers? What activities and situations would lead future teachers to put these ideas into action, to try them out, and so forth?

There are probably many ways to generate answers to these questions. For me, however, one way of addressing the issue was to create parallel courses to adapted mathematics courses that focus on problem-solving as Frédéric offers. But these courses are about teaching *through* problem-solving, where the reflection is geared as much toward mathematics itself as to the teaching of this mathematics through problem-solving. I am in fact currently developing and experimenting with a course like this, completely built along these ideas: a course in which students *experience* and explicitly *live* a variety of modalities for teaching through problem-solving in secondary schools (group work, mental mathematics, vertical surfaces, think-pair-share, games, etc.) by living it as students of mathematics and then reflecting on what happened and its potential for (teaching) mathematics. And then, each student would be asked to experiment with a number of these modalities (be it in micro-teaching situations or directly in their practicum). I believe that this has the potential to link directly to the kind of mathematical experiences that students live in their advanced and teacher-specific mathematics courses similar to those that Frédéric teaches. One thing, though, is that Frédéric and I do not teach in the same institution!

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